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On minimal neighbourhood-connected graphs

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Abstract

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The closed neighbourhood of a vertex u of a graph G is $u^* = \{v \mid v \text{ is adjacent to } u\} \cup \{u\}$. G is *neighbourhood-connected* if it is connected, and $G - u^*$ is connected but not complete, for all u in G. We consider neighbourhood-connected graphs G for which all $G - u^*$ are minimally k-connected, for $k = 1, 2$, and 3. In particular, we allow $G - u^*$ to be a cycle, wheel, or tree, and characterize the graphs G with this property.

1. Neighbourhood-connected graphs

We shall use the graph-theoretic notation of [1], so that a graph G has vertex set $V(G)$, edge set $E(G)$, and $\varepsilon(G)$ edges. If $U \subseteq V(G)$, then $G[U]$ denotes the subgraph induced by U. We write $uv \in E(G)$ to indicate that the pair $\{u, v\}$ forms an edge in G. We also use $u \to v$ to indicate that u is adjacent to v.

The graphs studied here arise in connection with *neighbourhood-connected* graphs. If $u \in V(G)$, then u^* denotes the *closed* neighbourhood of u, that is, $u^* := \{v \mid u \to v\} \cup \{u\}$. The neighbourhood of u is then $u^* - u$. A graph G is *neighbourhood-connected* (NC) if:

(1) G is connected; and

(2) $G - u^*$ is connected and not complete, for all $u \in V(G)$.

Complete graphs are excluded because they have the property that deleting any u^* in a complete graph G destroys all of G, which leads to uninteresting graphs.

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Neighbourhood-connected graphs have been studied in [2–6, 8] (where they are called 2-neighbourhood-connected). They arose in connection with the problem of constructing reliable communication networks, so that, should a node and all its immediate neighbours be destroyed, the remaining portion of the graph should still be connected. A graph is neighbourhood-k-connected (N k-C) if it is NC and $G-u^*$ is k-connected (see [1] for definitions of connectivity, k-connectedness, etc). In this paper we consider certain N k-C graphs in which the graphs $G-u^*$ are minimally k-connected. We call such a G *minimally neighbourhood k-connected* (min N k-C). In particular, we allow $G-u^*$ to be:

(i) a tree, which is a minimally 1-connected graph;

(ii) a cycle, which is minimally 2-connected, the smallest 2-connected graphs, in fact;

(iii) a wheel, which is minimally 3-connected.

We denote the complement of G by $\bar{G}$. Suppose that $G-u^* \cong X$. Notice that in $\bar{G}$, $\bar{G}[u^*-u] \cong \bar{X}$ (where u^* is now the neighbourhood in $\bar{G}$). We shall use this frequently. If all $G-u^*$ are isomorphic graphs, then in $\bar{G}$, all neighbourhood graphs $\bar{G}[u^*-u]$ are isomorphic. Such graphs $\bar{G}$ are called *homogeneous*, or graphs of *constant link* [7]. $\bar{G}[u^*-u]$ is called the *link* of $\bar{G}$. So the complement of a NC graph will sometimes be a graph of constant link. In particular, [7] classifies all graphs X with $\leqslant 6$ vertices as to whether X can be the constant link of some graph. It is also possible that $\bar{G}$ will sometimes be a disconnected graph. In such cases, we must consider each component of $\bar{G}$ separately. The vertices of each connected component of $\bar{G}$ induce a subgraph of G called a *connection unit*. G will then consist of one or more connection units such that each vertex u of G is joined to all vertices not in its connection unit.

2. Cycles and wheels

Suppose that we require $G-u^*$ to be a cycle. Since C_3, the 3-cycle, is also a complete graph, we consider only cycles C_m, where $m \geqslant 4$. The *line-graph* $L(X)$ of a graph X is the graph whose vertices are $E(X)$, with $a, b \in E(X)$ forming an edge ab in $L(X)$ if and only if a and b share a common end-point in X.

Theorem 2.1. *$G-u^* \cong C_4$ for all $u \in V(G)$, if, and only if, $\bar{G}$ is the line-graph of a 3-regular graph X without triangles.*

Proof. Consider $\bar{G}$. $\bar{G}[u^*-u]$ must be isomorphic to $\bar{C}_4$, which can also be denoted by $2K_2$. So $\bar{G}[u^*]$ is isomorphic to the graph of Fig. 1. It follows that every vertex u has degree 4 and is incident on exactly 2 triangles in $\bar{G}$. We now form a graph X from $\bar{G}$ as follows. $V(X)$ consists of the triangles of $\bar{G}$. Two triangles a and b are adjacent if and only if a and b share a common vertex in $\bar{G}$. Since a triangle has 3 vertices, X is 3-regular. X has no triangles, for otherwise

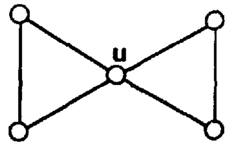

Fig. 1. $\bar{G}[u^* - u] \cong 2K_2$.

some $\bar{G}[u^*]$ would be incorrect. It is then easy to see that $\bar{G} \cong L(X)$. Notice that X may not be connected. In this case each component of $L(X)$ is the line-graph of a component of X.

It was proved in [7] that there are infinitely many graphs with constant link $2K_2$, but the relation with line-graphs does not seem to be indicated there. This shows that there are infinitely many NC graphs with $G - u^* \cong C_4$. The following two results follow from results proved in [7]. They can also be verified from first principles.

Lemma 2.2. *$G - u^* \cong C_5$ for all $u \in V(G)$, if and only if, each connected component of $\bar{G}$ is the graph of the icosahedron.*

Lemma 2.3. *$G - u^* \cong C_6$ for all $u \in V(G)$, if and only if, each connected component of $\bar{G}$ is isomorphic to $L(K_5)$ (and therefore each connection unit of G is a Petersen graph).*

Theorem 2.4. *There is no graph G for which all $G - u^* \cong C_m$, where $m \geq 7$.*

Proof. The proof is illustrated for $m = 7$, but the technique is general. Refer to Fig. 2. Suppose that $G - u^* \cong C_7$. Choose any v on this C_7, which is shaded in the diagram, and form the C_7 corresponding to $G - v^*$. No new edges among shaded vertices can be introduced, and none of them is adjacent to u. The only way to make $G - v^*$ into C_7 is to join u through 2 intermediate vertices to $G - u^*$, as shown. Now w is in both $G - u^*$ and $G - v^*$, so that no new edges can be added between w and any of the vertices shown in Fig. 2. But then deleting w^* from Fig. 2 leaves a configuration which cannot be a subgraph of any C_7. Therefore $G - w^*$ cannot be C_7. An identical argument holds for all $m \geq 7$. □

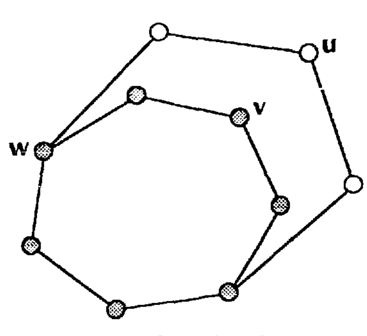

Fig. 2. $G - u^* \cong C_7$.

Lemmas 2.1 to 2.3, and Theorem 2.4 contain all NC-graphs G for which $G - u^* \cong C_m$ for any fixed m. Their complements are all graphs of constant link. In fact they basically contain all NC graphs in which all $G - u^*$ are isomorphic to cycles, of any assortment of lengths, as shown in Theorem 2.5 following.

Theorem 2.5. *If all $G - u^*$ are isomorphic to cycles, then in each connection unit of G, all the cycles $G - u^*$ have the same length.*

Proof. Choose any connection unit U of G, and let $G - u^* \cong C_r$, where r is the longest such cycle for any vertex $u \in U$. Notice that deleting u^* also deletes all connection units but U. Pick any v on this C_r, and consider $G - v^*$. It must be a cycle. As in Theorem 2.4, we find that u must be joined to the C_r through 2 intermediate vertices, as shown in Fig. 2. Therefore the cycle $G - v^*$ must have length $\geqslant r$, so that it is also isomorphic to C_r. Since $\bar{G}[U]$ is connected, all vertices $w \in U$ satisfy $G - w^* \cong C_r$. Thus, the connection unit U induces one of the graphs of Lemmas 2.1 to 2.3, and furthermore $r \leqslant 6$. If G has several connection units, each one is independent of the others, and may induce any of the graphs of Lemmas 2.1 to 2.3. □

The *wheel*, W_k, where $k \geqslant 3$, consists of a cycle C_k together with a vertex, called the *hub*, adjacent to all points of C_k. Wheels are minimally 3-connected graphs, since the removal of any edge leaves a graph with connectivity only 2. In fact, any 3-connected graph can be obtained from some wheel by the operations of splitting a vertex or adding an edge [9]. If $G - u^* \cong W_k$, then $\bar{G}[u^* - u] \cong \bar{C}_k + K_1$, that is, the disjoint union of an isolated point K_1 and the complement of a cycle C_k. Since $W_3 \cong K_4$, a complete graph, we consider only wheels W_k with $k \geqslant 4$. The 'union' theorem of [7] states that if G has constant link X and H has constant link Y, then the product $G \times H$ has constant link $X + Y$, the disjoint union of X and Y. In general, choose any vertex (u, v) of $G \times H$. Then

$$G \times H[(u, v)^* - (u, v)] \cong G[u^* - u] + H[v^* - v].$$

This gives the following.

Theorem 2.6. *Let $X - u^*$ be a cycle, for all $u \in V(X)$, and let $G \cong \overline{\bar{X} \times K_2}$. Then $G - u^*$ is a wheel, for all $u \in V(G)$.*

In particular, there are an infinite number of such G for W_4, and one only for each of W_5 and W_6, and none for W_k, where $k \geqslant 7$. However, this is not the only way to get wheels. It turns out to be a special case of the following technique.

Suppose that $G - u^*$ is a wheel W, with hub v, and let $G - v^*$ be a wheel W'. Then $W \cap W' = \emptyset$, since deleting v^* deletes all of W. Furthermore W' contains u. If u were not the hub of W', then W' would contain a vertex $w \not\rightarrow u$, which would force $w \in W \cap W'$, a contradiction. Therefore, we conclude that u is also the hub of W'. Thus to each vertex u of G, there corresponds a vertex v, the hub of $G-u^*$, and u is also the hub of $G-v^*$. Form the graph $G' := G + \{uv \mid u$ and v are corresponding hubs$\}$. Clearly each $G'-u^*$ is always a cycle, so each connection unit of G' will be one of the graphs of Lemmas 2.1 to 2.3. Now the connection units of G' are the connected components of $\bar{G}'$, which are formed from those of $\bar{G}$ by removing a set of edges forming a perfect matching. This shows us how to construct G from the connection units described in Lemmas 2.1 to 2.3:

(1) Choose a number of connected graphs $X_1, X_2, \ldots, X_n$, where each X_i is one of $L(K_5)$, the icosahedron, or the line-graph of a 3-regular graph without triangles, and take their disjoint union.

(2) For each $u \in X_i$, choose a vertex v in some X_j, where $i \neq j$, such that a *pairing* of all the vertices is obtained.

(3) Define G to be the complement of the graph so-obtained.

Theorem 2.7. *The G constructed has all $G-u^*$ isomorphic to wheels. Every such G can be built in this way.*

Proof. Consider $G-u^*$, where $u \in X_i$ is paired with $v \in X_j$. Deleting u^* will delete all X_k, where $k \neq i, j$. It also deletes all of X_j except the vertex v. The only part of X_i not deleted forms a cycle, by Lemma 2.1, 2.2, or 2.3. v is joined to this cycle, thereby becoming the hub of a wheel. So every $G-u^*$ is a wheel. The above discussion shows that this includes all such graphs. □

Notice that the wheels $G-u^*$ and $G-v^*$ corresponding to paired vertices (corresponding hubs), need not have the same order, since in forming G, we are free to pair the vertices in any way which yields a global pairing of all vertices.

3. Trees

Trees are minimally connected graphs, for deleting any edge disconnects them. We want to determine NC graphs G for which all $G-u^*$ are trees. Since $G-u^*$ must not be complete, we require trees with at least 3 vertices. If $G \cong C_m$, for some $m \geq 6$, then every $G-u^*$ is a path of length $m-4$, which is a tree. Henceforth we assume that all $G-u^*$ are trees with ≥ 3 vertices.

The *girth* of G is $\gamma(G)$, the length of a shortest cycle in G. We divide the characterization of the graphs G into cases depending on γ.

Lemma 3.1. *Let C be any cycle in G. Then every $u \in V(G)$ is adjacent to some vertex of C. Furthermore, if $\gamma \geq 5$, then each $u \notin C$ is adjacent to exactly one vertex of C.*

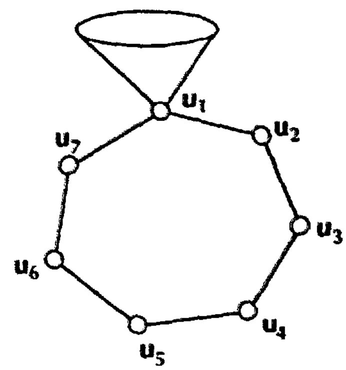

Fig. 3. $\gamma(G) \geq 7$.

Proof. If some u were not adjacent to any point of C, then $G - u^*$ must contain C, which is impossible, since $G - u^*$ is a tree. If $\gamma \geq 5$, and $u \notin C$ were adjacent to two vertices x and y of C, then u, together with one of the segments of C defined by x and y, would create a cycle of length at most $2 + \gamma/2$, which is less than γ, a contradiction. □

Theorem 3.2. *If $\gamma(G) \geq 7$, then only $G \cong C_m$ is possible.*

Proof. Suppose that $\gamma(G) = m \geq 7$, and let $C = (u_1, u_2, \ldots, u_m)$ be an m-cycle in G. This is illustrated in Fig. 3 for $m = 7$. Since $G \not\cong C_m$, some vertex $v \notin C$. Suppose $v \rightarrow u_1$. By Lemma 3.1, v is adjacent only to u_1 on C. Let S_i denote the set of vertices not on C adjacent to u_i, for $i = 1, 2, \ldots, m$. v must be adjacent to a vertex $w \in S_i$, for some i, otherwise $G - u_2^*$ would be disconnected. But this would create a cycle of length $\leq 3 + m/2 < m$. So if $\gamma = m \geq 7$, then $G \cong C_m$ is the only NC graph for which each $G - u^*$ is a tree. □

Let DS(k) denote a *double k-star*. It is formed from two k-stars by joining the corresponding vertices of degree 1, thereby increasing their degree to 2. This is illustrated in Fig. 4. DS(k) has $2k + 2$ vertices.

Theorem 3.3 *If $\gamma(G) = 6$ and $G \not\cong C_6$, then $G \cong$ DS(k), for some k.*

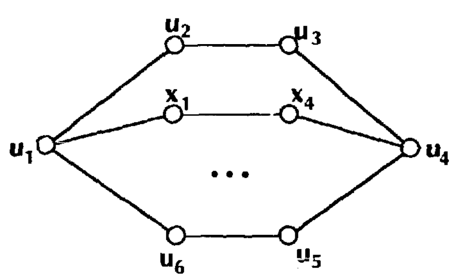

Fig. 4. $DS(k)$.

Proof. Let $C=(u_1, u_2, \ldots, u_6)$ be a 6-cycle in G (see Fig. 4). By Lemma 3.1, all $u \notin C$ are adjacent to C. Note that u can be adjacent to only one $u_i \in C$, for otherwise $\gamma<6$. Let S_i be the set of vertices not on C which are adjacent to u_i.

Not all S_i are empty, so without loss of generality, pick $x_1 \in S_1$. x_1 must be adjacent to some $x_j \in S_j$, for some j; otherwise in $G-u_2^*$, x_1 and u_4 would be separated. Only $j=4$ is possible, since $\gamma=6$. So $x_1 \in S_1$ is adjacent to $x_4 \in S_4$. Similarly, each vertex of S_1 must be adjacent to some vertex of S_4. No two vertices of S_1 can be joined to the same vertex of S_4, or a 4-cycle would be created.

Finally, the remaining sets S_2, S_3, S_5, and S_6 must all be empty. For if S_2, say, contained a vertex y, then $G-y^*$ would contain the 6-cycle $(u_1, x_1, x_4, u_4, u_5, u_6)$. Thus, $G \cong \mathrm{DS}(k)$, where $k=|S_1|+2$. □

Theorem 3.4. *If $\gamma(G)=5$ and $G \not\cong C_5$, then G is the graph shown in Fig. 5.*

Proof. Let $C=(u_1, u_2, \ldots, u_5)$ be a 5-cycle in G (see Fig. 5). By Lemma 3.1, each $u \notin C$ is joined to exactly one $u_i \in C$. Let S_i denote the set of all $u \notin C$ adjacent to u_i. Not all S_i are empty, so pick $x_1 \in S_1$. x_1 must be adjacent to a vertex of some S_i, where $i \neq 1$, for otherwise $G-u_2^*$ would be disconnected. We cannot have $i=2$ or 5, for this would create a 4-cycle. Without loss of generality, pick $x_1 \rightarrow x_3 \in S_3$. If there are no other vertices or edges in G, then in $G-u_2^*$, x_1 and x_3 are separated from u_4 and u_5. So at least one of S_4 and S_5 is non-empty, and either $x_1 \rightarrow x_4 \in S_4$ or $x_3 \rightarrow x_5 \in S_5$. By symmetry, we may choose $x_1 \rightarrow x_4$. This gives the graph of Fig. 5. We must still show that this is the entire graph. We first show that $S_2=S_5=\emptyset$. For suppose that $x_2 \in S_2$, say. Then $x_2 \not\rightarrow x_1$ and $x_2 \not\rightarrow x_3$, so that $G-x_2^*$ must contain the cycle $(u_1, x_1, x_3, u_3, u_4, u_5)$, a contradiction. Finally, we show that $|S_1|=|S_3|=|S_4|=1$. If x_1 were adjacent to another $x \in S_3$, this would create a 4-cycle (x_1, x, u_3, x_3), which is impossible. If there were $y_1 \in S_1$ and $y_3 \in S_3$, where $y_1 \neq x_1$ and $y_3 \neq x_3$, and $y_1 \rightarrow y_3$, then $G-x_4^*$ would contain the cycle $(y_1, u_1, u_2, u_3, y_3)$, a contradiction. Hence, G is the unique graph depicted in Fig. 5. □

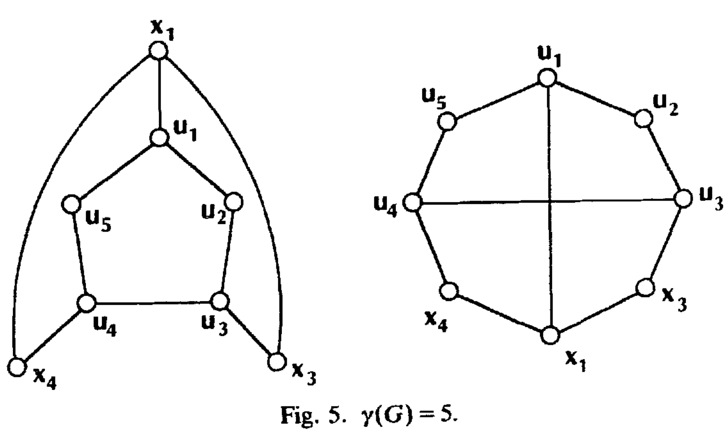

Fig. 5. $\gamma(G)=5$.

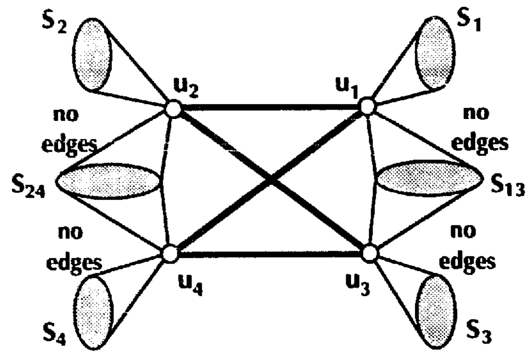

Fig. 6. $\gamma(G)=4$.

Notice that this graph is hamiltonian. When redrawn to illustrate the hamilton cycle, it is seen to be an 8-cycle with two opposite diagonals. Both drawings are shown in Fig. 5.

Suppose now that $\gamma(G)=4$. Choose a 4-cycle (u_1, u_2, u_3, u_4). There can be no vertex joined to both u_i and u_{i+1}, for this would create a triangle; however u_i and u_{i+2} are acceptable. Let S_i denote the set of vertices not on C which are adjacent to u_i, but to no other u_j. Let S_{ij} denote the set of vertices not on C which are adjacent to both u_i and u_j. This is illustrated in Fig. 6.

There are no edges within the shaded sets, for they would create triangles. Similarly, there can be no edges between the sets S_i and S_{ij} or between S_i and S_{ji}, as these would also create triangles. This is indicated in the diagram. Notice that $S_i \neq \emptyset$, for $i=1,2,3,4$. For if some $S_i=\emptyset$, then in $G-u_{i+2}^*$, u_i would be an isolated vertex, which is impossible.

The following lemmas are useful in proving our result about the girth 4 case. We will frequently require expressions like u_{i+1} or S_{i+2}, etc, where $i \in \{1,2,3,4\}$. In cases like this the arithmetic is to be computed as the unique integer, modulo 4, in the set $\{1,2,3,4\}$.

Lemma 3.5. *Each vertex of $S_{i,\,i+2}$ is joined to exactly one vertex of S_{i-1} and one vertex of S_{i+1}, where $i=1,2$.*

Proof. Let G be as illustrated in Fig. 6. Form $G-u_1^*$. It contains u_3, S_3, S_2, S_4, and S_{24}. Since there are no edges within S_{24}, there must be some edges $[S_{24}, S_3]$ or S_{24} would be a set of isolated vertices in $G-u_1^*$. Therefore each vertex of S_{24} must be joined to at least one vertex of S_3. If some vertex were joined to more than one, then this would create a cycle through u_3 in $G-u_1^*$. So each vertex is joined to exactly one. Similarly, each vertex of S_{24} must be joined to exactly one vertex of S_1. A similar argument works for S_{13}.

Lemma 3.6. *For each i, and each $x_i \in S_i$, at least one of the following hold*:

(i) *x_i is adjacent to an element of S_{i-1} and an element of S_{i+1}*;

(ii) *x_i is adjacent to an element of S_{i+2}.*

Proof. Suppose that x_i is not adjacent to any element of S_{i+2}. In $G-u^*_{i+1}$, x_i would be isolated if not adjacent to an element of S_{i-1}. Similarly, in $G-u^*_{i-1}$, x_i would be isolated if not adjacent to an element of S_{i+1}. So if x_i is not adjacent to any element of S_{i+2} then it is adjacent to a vertex in each of S_{i-1} and S_{i+1}. □

Lemma 3.7. *Each $x_i \in S_i$ can have at most one neighbour in S_{i+1} and at most one neighbour in S_{i-1}, for $i=1,2,3,4$.*

Proof. Assume that some $x_i \in S_i$ had two neighbours, y_1 and y_2 in S_{i+1}. Then $G-u^*_{i-1}$ would contain a cycle, namely (x_i, y_1, u_{i+1}, y_2). Similarly, if $y_1, y_2 \in S_{i-1}$, then $G-u^*_{i+1}$ would contain a cycle. □

Lemma 3.8. *If $x_i \in S_i$ has a neighbour in both S_{i-1} and S_{i+1}, then x_i is the only member of S_i.*

Proof. Suppose that $x_i \in S_i$ had a neighbour $y_1 \in S_{i-1}$ and $y_2 \in S_{i+1}$. Assume that S_i also contains $x \neq x_i$. By Lemma 3.7, x cannot be adjacent to y_1 or y_2. But then $G-x^*$ has a cycle, namely $(x_i, y_1, u_{i-1}, u_{i+2}, u_{i+1}, y_2)$ □

Lemma 3.9. *If some $x_i \in S_i$ does not have a neighbour in both S_{i-1} and S_{i+1}, then $S_{i,i+2}$ is empty.*

Proof. Suppose that $x_i \in S_i$ does not have a neighbour in both of S_{i-1} and S_{i+1}. Then $G-x_i^*$ would contain all of one of S_{i-1} or S_{i+1}, say S_{i-1}. By Lemma 3.5, each element $z \in S_{i,i+2}$ is adjacent to some element, say y, of S_{i-1}. But then (z, y, u_{i-1}, u_{i+2}) is a cycle, so that $S_{i,i+2}$ must be empty. □

We now proceed to show that each S_i has exactly one element.

Lemma 3.10. *For each i, $|S_i|=1$.*

Proof. *Case* 1: *S_i and S_{i+1} both contain more than one element, for some i.*

Suppose that some $x \in S_i$ has a neighbour $y \in S_{i+1}$. Pick $z \neq x$ in S_i. Since $|S_i|>1$, z cannot have a neighbour in both S_{i-1} and S_{i+1}, by Lemma 3.8. Therefore z must be adjacent to some $w \in S_{i+2}$, by Lemma 3.6. Now $y \nrightarrow w$, by Lemma 3.8, since $|S_{i+1}|>1$. It follows that $G-y^*$ contains the cycle $(u_i, z, w, u_{i+2}, u_{i-1})$, which is impossible. Hence, there are no edges between S_i and S_{i+1}.

Now x must have some neighbour $v \in S_{i+2}$, by Lemma 3.6. If $y \in S_{i+1} \nrightarrow v$, then $G-y^*$ contains the cycle $(u_i, x, v, u_{i+2}, u_{i-1})$. Hence each $y \in S_{i+2}$ is joined to v. But since $|S_{i+1}|>1$, this means that $v \in S_{i+2}$ is adjacent to more than one element of S_{i+1}, contradicting Lemma 3.7.

So S_i and S_{i+1} cannot both contain more than one element.

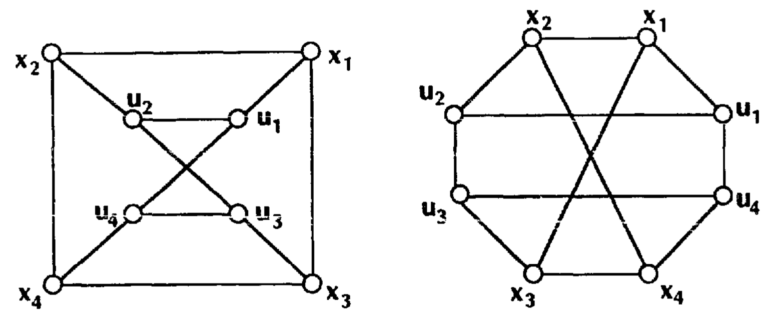

Fig. 7. G_8.

Case 2: *S_i contains more than one element, for some i.*

Notice that $|S_{i-1}|=|S_{i+1}|=1$, by Case 1. Let $S_{i-1}=\{x_{i-1}\}$ and $S_{i+1}=\{x_{i+1}\}$. Suppose that x_{i+1} has no neighbour in S_i. Then by Lemmas 3.6 and 3.7, $x_{i+1}\rightarrow x_{i-1}$ and x_{i-1} is adjacent to exactly one $v\in S_i$. Pick $w\neq v$ in S_i. Then $G-w^*$ contains the cycle $(x_{i+1}, x_{i-1}, u_{i-1}, u_{i+2}, u_{i+1})$. So x_{i+1} must be adjacent to $x_i\in S_i$. Similarly, x_{i-1} must be adjacent to some $y\in S_i$ (maybe $y=x_i$). Now $x_{i+1}\not\rightarrow x_{i-1}$, or $G-u_{i+2}^*$ would contain the cycle $(x_{i+1}, x_{i-1}, y, u_i, x_i)$. If $y\neq x_i$, then $G-x_{i+1}^*$ contains the cycle $(x_{i-1}, y, u_i, u_{i-1})$. So it must be that $y=x_i$. But $|S_i|>1$, so pick $z\neq x_i$ in S_i. Then $G-z^*$ contains the cycle $(x_{i-1}, x_i, x_{i+1}, u_{i+1}, u_{i+2}, u_{i-1})$, again a contradiction.

Thus, no S_i can contain more than one element. It follows from Lemma 3.5 that each $|S_i|=1$. □

We are now in a position to characterise all graphs of girth 4 of the desired type. Let G_8 be the graph shown in Fig. 7.

Theorem 3.11. *If $\gamma(G)=4$, then G is one of G_8 or $\overline{K_m\times K_2}$, for some $m\geqslant 4$.*

Proof. By Lemma 3.10, we can take $S_i=\{x_i\}$.

Case 1: *$x_i\not\rightarrow x_{i+2}$, for some i.*

By Lemma 3.6, we must have $x_{i+2}\rightarrow x_{i-1}\rightarrow x_i$, and by Lemma 3.6 we must have $x_{i+2}\rightarrow x_{i+1}\rightarrow x_i$. It follows that $x_{i-1}\not\rightarrow x_{i+1}$, or a triangle (x_i, x_{i+2}, x_{i+1}) would be created, which is impossible, since $\gamma=4$. If $S_{i,i+2}\neq\emptyset$, pick $v\in S_{i,i+2}$. By Lemma 3.5, v is adjacent to x_{i-1} and x_{i+1}. $G-v^*$ contains $x_i, x_{i+2}, u_{i-1}, u_{i+1}, S_{i,i+2}-v$, and possibly some of $S_{i-1,i+1}$. This can only be connected if $G-v^*$ contains exactly one $w\in S_{i-1,i+1}$, which must be adjacent to x_i and x_{i+2}, by Lemma 3.5. Furthermore, w must also be adjacent to each vertex of $S_{i,i+2}-v$. It follows that for each $v\in S_{i,i+2}$, v is adjacent to every vertex of $S_{i-1,i+1}$, but one. Hence, $|S_{i,i+2}|=|S_{i-1,i+1}|$. This completely determines the structure of G, since we now know the adjacencies between any two sets S_i, S_j, and S_{ij}, and the vertices u_i, where $i, j=1, 2, 3, 4$.

The graph G produced is bipartite. For let $X=\{u_1, u_3, x_2, x_4\}\cup S_{24}$, and $Y=\{u_2, u_4, x_1, x_3\}\cup S_{13}$. (X, Y) is a bipartition of G. Furthermore, for each $x\in X$, x is adjacent to all of Y, except for one $y\in Y$. G is almost a complete

Table 1

girth	graph	$G-u^*$
4	$\overline{K_m \times K_2}$, $m \geq 4$	S_{m-1}
4	G_8	P_3
5	G_5	P_3 and P_4
6	$DS(k)$, $k \geq 3$	S_k and S_{k-1}^+
$k \geq 6$	C_k	P_{k-4}

bipartite graph $K_{m,m}$, where $m = 4 + |S_{24}|$. The complement of G consists of two complete graphs K_m, joined by a perfect matching, that is $\bar{G} \cong K_m \times K_2$, giving $G \cong \overline{K_m \times K_2}$, where $m \geq 4$.

Case 2: $x_i \to x_{i+2}$, *for all i.*

By Lemma 3.9, $S_{i,i+2} = S_{i-1,i+1} = \emptyset$. By Lemma 3.6, either $x_1 \to x_2$ or $x_1 \to x_4$, but not both, for that would create a triangle. Without loss of generality, take $x_1 \to x_2$. Now $x_3 \not\to x_2$, for that would also create a triangle, so by Lemma 3.6, $x_3 \to x_4$. Hence, the only possible graph is G_8 of Fig. 7, which has been drawn twice in order to illustrate the hamilton cycle. □

We have determined all neighbourhood-connected graphs G for which each $G - u^*$ is a tree, and $\gamma(G) \geq 4$. The case of $\gamma = 3$ appears to be quite difficult, requiring a very detailed case-by-case analysis. We have not determined all girth 3 graphs with this property, but they appear to be numerous.

It is interesting to note which trees can appear as $G - u^*$. We denote by P_k a path with k edges. S_k denotes the k-star, that is, the complete bipartite graph $K_{1,k}$. The *extended k-star* S_k^+ is obtained by adding k new vertices to S_k, each joined to one of the vertices of degree 1, thereby giving them degree 2 in S_k^+. Surprisingly, only 3-paths, 4-paths, stars, and extended stars are allowed. This is summarised in Table 1.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Elsevier, Amsterdam, 1976).

[2] G. Gunther, Minimum k-neighbour-connected graphs and affine flags, Ars Combin. 26 (1988) 83–90.

[3] G. Gunther and B.L. Hartnell, Flags and neighbour-connectivity, Ars Combin. 24 (1987) 31–38.

[4] G. Gunther, B.L. Hartnell and R. Nowakowski, Neighbour-connected graphs and projective planes, Networks 17 (1987) 241–247.

[5] G. Gunther, Neighbour-connectivity in regular graphs, Discrete Appl. Math. 77 (1985) 233–243.

[6] G. Gunther and B.L. Hartnell, Optimal k-secure graphs, Discrete Appl. Math. 2 (1980) 225–231.

[7] J.I. Hall, Graphs of constant link and small degree or order, J. Graph Theory 8 (1985) 419–444.

[8] A. Meir and J. Moon, Survival under random coverings of trees, Graphs Combin. 4 (1988) 49–65.

[9] W.T. Tutte, A theory of 3-connected graphs, Konink. Nederl. Akad. van W. Proc. 64 (1961) 441–445, reprinted in: D. McCarthy and R. Stanton, eds., Selected Papers of W.T. Tutte (CBRC, Winnipeg, Man., Canada, 1979).